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CONTACT PROBLEM FOR A SEMI-INFINITE CYLINDRICAL SHELL

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The problem of the impression of pointed stamps along segments of the cross-sectional circle of a semi-infinite cylindrical shell supported freely at the endface is considered. The edges of the stamps are absolutely stiff, of constant radius, and have no sharp angles. The influence of the shell endface on the character of the change in reaction of the stamps is investigated. The problem is solved on the basis of the shell theory equations constructed taking account of the Kirchoff-Love hypothesis. The friction between the shell surface and the stamp edges is not taken into account.

1. Let us consider a semi-infinite cylindrical shell (Fig. 1), freely supported on the endface $\xi = 0$ compressed along segments of the circle $\xi = \xi_0$ by identical stamps, where m denotes the number of stamps ($(m = 2)$ in Fig. 1).

We consider the stamp edges to be sharp and absolutely stiff so that the contact between the shell and stamp is on the arc of a circle whose magnitude is characterized by the central angle θ to be determined. We consider the curvature $1/R_1$ of the stamp edges to be constant. Linear stress resultants q (reactions) act from the stamp on the shell, and we consider them directed along the normal to the surface within the shell, without taking account of friction. Proceeding from the linear theory of thin shallow shells, we shall also assume that either the angle θ is small, or the radius R_1 of the stamp edges differs slightly from the radius of the outer surface R_0 of the shell.

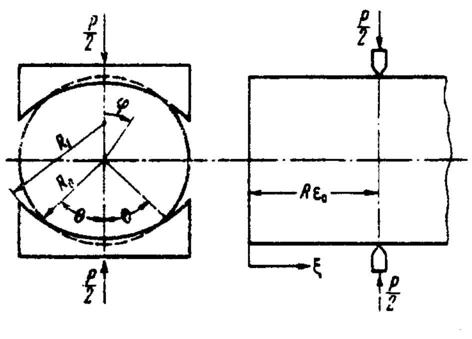

Fig. 1.

We obtain the initial equation of the problem from the condition of complete abutment of the shell to the stamp in the contact zone, which can be written as $\varkappa_2 = 1/R_1 - 1/R_0$, where $\varkappa_2$ is the bending strain of the shell in the circumferential direction on the line of contact. Knowing the Green's function Ψ for a semi-infinite shell freely supported on the endface $\xi = 0$ the strain $\varkappa_2$ can be determined by formulas from [1]. Let us show that

$$\Psi = \Phi(\xi - \xi_0,\ \varphi - \varphi_0) - \Phi(\xi + \xi_0,\ \varphi - \varphi_0)$$

where $\Phi(x,\ y)$ is the Green's function for an infinite shell, ξ_0, φ_0 is the point of application of a concentrated factor, ξ, φ is the point where there is a solution, ξ, ξ_0 are longitudinal coordinates referred to the radius, and φ, φ_0 are transverse coordinates given by the central angles (Fig. 1.) The function $\Phi(\xi - \xi_0,\ \varphi - \varphi_0)$ is a particular solution of the equation [1]

$$D\Phi = \delta(\xi - \xi_0,\ \varphi - \varphi_0) \tag{1.1}$$

where D is a known linear differential operator, $\delta(\xi - \xi_0,\ \varphi - \varphi_0)$ is the two-dimensional delta function. Hence, for the domain $\xi \geqslant 0$, $\xi_0 > 0$ the function $\Phi(\xi + \xi_0,\ \varphi - \varphi_0)$ is a solution of the homogeneous equation $D\Phi = 0$. Therefore, the function Ψ defined above is a solution of (1.1). On the other hand, because of the evenness of the function $\Phi(x,\ y)$ the function Ψ and all its even derivatives vanish on the line $\xi = 0$. Only even derivatives of Ψ appear in the boundary conditions for a shell freely supported on the endface $\xi = 0$. Hence, the assertion made can be considered proved.

Utilizing the results of [1], and omitting intermediate computations, the initial integral equation to determine the stamp reaction q are written as

$$\int_{-\beta}^{\beta} \ln\left|2\sin\frac{\alpha - \alpha_0}{2}\right| q\,d\alpha_0 = \int_{-\beta}^{\beta}\{K(\alpha - \alpha_1) - K_1(\alpha - \alpha_1)\}\, q\,d\alpha_1 - m\omega_0 \tag{1.2}$$

$$K_1(\alpha - \alpha_1) = \frac{x \operatorname{sh} 2x}{\operatorname{ch} 2x - \cos(\alpha - \alpha_1)} - \frac{1}{2}\ln\left[2(\operatorname{ch} 2x - \cos(\alpha - \alpha_1))\right]$$

$$K(\alpha - \alpha_1) = \sum_{k=1}^{\infty} \frac{b_k}{k}\cos k(\alpha - \alpha_1) \tag{1.3}$$

$$\omega_0 = \frac{4\pi E h a^2}{1 - \nu^2}\left(1 - \frac{R_0}{R_1}\right), \qquad a^2 = \frac{h^2}{12R^2}, \quad \alpha = m\varphi, \quad \beta = m\theta \tag{1.4}$$

$$b_k = -1 + (1 + 2kx)\, e^{-2kx} + \sum_{j=1}^{2} \frac{1}{r_j}\left[c_j(1 - \eta_j) - d_j\zeta_j\right] \tag{1.5}$$

$$x = m\xi_0, \quad \eta_j + i\zeta_j = e^{-2kx(-q_j + ip_j)}, \quad r_j e^{i\gamma_j} = p_j q_j (b_j + i a_j) \tag{1.6}$$

Here E, ν is the elastic modulus and Poisson's ratio; R, h are the radius of the middle surface and the shell thickness; p_j, q_j the real and imaginary parts, respectively, of the roots of the chracteristic equation of shell theory whose value is given in [1] for shallow shells, and a_j, b_j are constant coefficients also presented in [1]. The coefficients c_j and d_j are, respectively, obtained if the quantity $n^{k-6}(-1)^k \rho_j^{\,k}$ multiplied by $\sin(\gamma_j - k\omega_j)$ and $\cos(\gamma_j - k\omega_j)$, is substituted into the operator

$$-\left\{(2 + \nu)\frac{\partial^4(\cdot)}{\partial\xi^2\partial\varphi^2} + \frac{\partial^4(\cdot)}{\partial\varphi^4} + \frac{\partial^2}{\partial\varphi^2}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\varphi^2}\right)^2(\cdot)\right\} \tag{1.7}$$

in place of $\partial^k(\cdot)/\partial\xi^k$ and the quantity n^m sign $(d^m \cos\varphi / d\varphi^m)$ in place of $\partial^m(\cdot)/\partial\varphi^m$, where $n = km$ and ρ_j and ω_j denote the modulus and argument of the complex number $q_j + ip_j$.

The differential operator (1.7) is the operator used in the fundamental resolving Green's function Ψ to obtain the Green's function characterizing the bending strain of the shell $\varkappa_2$.

A kernel of logarithmic type is obtained in the left side of (1.2) by extracting the principal value of the kernel from the Green's function $\Phi(\xi - \xi_0, \varphi - \varphi_0)$ for an infinite shell. The kernel K_1 in the right side is the principal value of the kernel of the Green's function $\Phi(\xi + \xi_0, \varphi - \varphi_0)$ and $K(\alpha - \alpha_1)$ is the part of the kernel remaining after extraction of the principal value. It can be shown that the coefficients b_k of the kernel (1.3) decrease as $1/k^4$. The kernel K_1 could also not be extracted in closed form since neither the function $\Phi(\xi + \xi_0, \varphi - \varphi_0)$ itself, nor its derivatives vanish at infinity for $\xi = \xi_0$, $\varphi = \varphi_0$ and $\xi_0 \neq 0$. But for small ξ_0 the kernel K_1 can become very large according to (1.3), hence without extracting K_1 the series for the kernel K would converge quite slowly.

The solution of (1.2) for $\omega_0 = \mathrm{const}$ will evidently be even. It must be subjected to the condition

$$\int_{-\beta}^{\beta} q \cos \frac{\alpha}{m} d\alpha = \frac{P}{R} \qquad (1.8)$$

Here P / m is the force applied to the stamp from outside (Fig. 1). Condition (1.8) establishes a relation between the angle $\beta = m\theta$, characterizing the magnitude of the contact zone and the force P. In order to isolate the singularity of the solution q at the ends of the zone of contact explicitly and to simplify the subsequent solution of the problem, let us convert (1.2) to a Fredholm equation of the second kind.

2. Let us consider the equation

$$\int_{-\beta}^{\beta} \ln \left| 2 \sin \frac{\alpha - \alpha_0}{2} \right| q d\alpha = f(\alpha_0) \qquad (2.1)$$

with a known right side which we consider a continuous and even function. Integrating the left side of (2.1) by parts, let us reduce it to a singular integral equation with kernel of the type $\operatorname{ctg} {}^1/_2 (\alpha - \alpha_0)$ relative to the function Q defined by the equation $dQ / d\alpha = q$ which is reduced to an equation with Cauchy kernel by means of the change of variable $t = \exp(i\alpha)$. Having obtained a bounded solution Q (because of physical considerations), and having differentiated it, we find the solution of (2.1) in the form

$$q(\alpha_0) = \frac{1}{2\pi^2 X(\alpha_0)} \int_{-\beta}^{\beta} \frac{X(\alpha) f'(\alpha) d\alpha}{\sin {}^1/_2 (\alpha - \alpha_0)} + \frac{A}{\pi} \frac{\cos {}^1/_2 \alpha_0}{X(\alpha_0)} \qquad (2.2)$$

The constant A is determined from the condition

$$A \ln \sin \frac{\beta}{2} = \frac{1}{\pi} \int_{-\beta}^{\beta} \frac{f(\alpha)}{X(\alpha)} \cos \frac{\alpha}{2} d\alpha \qquad (2.3)$$

The canonical function

$$X(\alpha) = \sqrt{2(\cos \alpha - \cos \beta)} \qquad (2.4)$$

Condition (2.3) is the condition for the existence of a bounded solution Q to the intermediate singular equation. The solution Q of an equation with cot type kernel could also be obtained by the method of Sedov [2]. Equations of this kind are in the class of equations with automorphic kernels [3], however, it is not expedient to apply this theory in this particular case. If the right side in (1.2) is provisionally considered a known function, then it agrees with (2.1). Hence, substituting the right side of (1.2) into the solution (2.2) and condition (2.3) in place of $f(\alpha)$, we transform them after interchanging the order of integration and evaluating the inner integrals, to the form

$$q(\alpha_0) + \frac{1}{\pi X(\alpha_0)} \int_{-\beta}^{\beta} [R(\alpha_1, \alpha_0) - R_1(\alpha_1, \alpha_0)]\, q\, d\alpha_1 + \frac{A}{\pi} \frac{\cos {}^1/_2 \alpha_0}{X(\alpha_0)} \qquad (2.5)$$

$$A \ln \sin \frac{\beta}{2} = \int_{-\beta}^{\beta} [\Psi(\alpha_1) - \Psi_1(\alpha_1)]\, q\, d\alpha_1 - \omega_0 m \qquad (2.6)$$

$$R(\alpha_1, \alpha_0) = \sum_{k=1}^{\infty} b_k \cos k\alpha_1 J_k(\alpha_0) \qquad (2.7)$$

$$\Psi(\alpha_1) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{b_k}{k} (P_k - P_{k-1}) \cos k\alpha_1 \qquad (2.8)$$

Here $P_k = P_k(\cos \beta)$ are Legendre polynomials which are evaluated by utilizing the integral representation [4]

$$\int_0 \frac{\cos(k + {}^1/_2)\alpha\, d\alpha}{\sqrt{2(\cos\alpha - \cos\beta)}} = \frac{\pi}{2} P_k(\cos\beta)$$

The function $J_k(\alpha_0)$ in (2.7) is

$$J_k(\alpha_0) = \frac{1}{2\pi} \int_{-\beta}^{\beta} \frac{X(\alpha) \sin k\alpha d\alpha}{\sin {}^1/_2 (\alpha - \alpha_0)} = \sum_{m=0}^{k} a_m \cos(k - m + {}^1/_2)\alpha_0 \qquad (2.9)$$

$$a_0 = 1, \qquad a_1 = -\cos\beta, \qquad a_m = P_m - 2\cos\beta\, P_{m-1} + P_{m-2} \qquad (2.10)$$

The integrals (2.9) were evaluated by residue theory [5] by passage to the variable $t = \exp(i\alpha)$

The kernel $R_1(\alpha_1, \alpha_0)$ in (2.5) has the form

$$R_1(\alpha_1, \alpha_0) = -\frac{1}{2\pi} \int_{-\beta}^{\beta} \frac{dK_1(\alpha - \alpha_1)}{d\alpha} \frac{X(\alpha)\, d\alpha}{\sin {}^1/_2 (\alpha - \alpha_0)} \qquad (2.11)$$

where K_1 is the kernel (1.3). Evaluating the integral in the right side of (2.11), we obtain

$$R_1(\alpha_1, \alpha_0) = -\cos\frac{\alpha_0}{2} + \frac{\rho_2}{2\rho_1} \cos(\varphi_1 + \varphi_2) + \frac{x}{\rho_1\rho_2} \Big\{ \frac{\rho_2}{\rho_1} (\cos\alpha_0 - \cos\beta) \times$$
$$\times \Big[\Big(1 + \frac{\cos\omega}{\rho_1^2}\Big) \cos(\varphi_2 - \varphi_3) - \frac{1}{\rho_1^2} \cos(\varphi_2 + \varphi_3) \Big] - \sin(\varphi_1 - \varphi_2) \Big\} \qquad (2.12)$$

where

$$\rho_1 e^{i\varphi_1} = \operatorname{sh} x \cos {}^1/_2\,\omega + i \operatorname{ch} x \sin {}^1/_2\,\omega \tag{2.13}$$
$$\rho_2{}^2 e^{2i\varphi_2} = 2(\operatorname{ch} 2x \cos\alpha_1 - \cos\beta + i \operatorname{sh} 2x \sin\alpha_1)$$
$$\rho_3 e^{i\varphi_3} = \operatorname{ch} x \cos {}^1/_2\,\omega + i \operatorname{sh} x \sin {}^1/_2\,\omega, \qquad \omega = \alpha_0 - \alpha_1$$

The function Ψ_1 in (2.6) is

$$\Psi_1(\alpha_1) = \frac{1}{\pi}\int_{-\beta}^{\beta} \frac{K_1(\alpha - \alpha_1)\cos {}^1/_2\,\alpha}{X(\alpha)}\, d\alpha \tag{2.14}$$

This integral has not been evaluated successfully in closed form. It can be taken either numerically (as will be discussed below), or by using the expansion

$$K_1(\alpha - \alpha_1) = \sum_{k=1}^{\infty} \frac{1 + 2kx}{k} e^{-2kx} \cos k(\alpha - \alpha_1)$$

for the kernel (1.3). Then

$$\Psi_1(\alpha_1) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1 + 2kx}{k} e^{-2kx}(P_k + P_{k-1}) \cos k\alpha_1 \tag{2.15}$$

As is proved in the theory of singular integral equations [6], the Carleman method used above to obtain (2.5) is equivalent. Hence, taking into account the uniqueness of the bounded solution of (2.1), the deduction can be made that (1.2) contains as many solutions as does (2.5). This latter has a unique solution, not bounded at the endpoints, with a singularity of the type $1 / X(\alpha)$, where $X(\alpha)$ is the canonical function (2.4). This solution is determined if the magnitude $\beta = m\theta$ of the contact zone is given, which is related to the external stress resultant P by the condition (1.8).

3. We seek the solution of (2.5) in the form

$$q = \frac{Ay(\alpha)}{\pi X(\alpha)} \tag{3.1}$$

Then Eq. (2.5), conditions (2.6) and (1.8) are transformed into the following relationships

$$y(\alpha_0) + \frac{1}{\pi}\int_{-\beta}^{\beta} \{R(\alpha_1, \alpha_0) - R_1(\alpha_1, \alpha_0)\} \frac{y\,d\alpha_1}{X(\alpha_1)} = \cos\frac{\alpha_0}{2} \tag{3.2}$$

$$A = -\omega_0 m\left[\ln\sin\frac{\beta}{2} - \frac{1}{\pi}\int_{-\beta}^{\beta} \frac{\Psi(\alpha_1) - \Psi_1(\alpha_1)}{X(\alpha_1)} y\,d\alpha_1\right]^{-1} \tag{3.3}$$

$$\frac{P}{R} = \frac{A}{\pi}\int_{-\beta}^{\beta} \frac{y(\alpha)\cos\dfrac{\alpha}{m}}{X(\alpha)}\, d\alpha \tag{3.4}$$

Let us transfer to the new variable γ in (3.2) – (3.4), defined by the formula

$$\sin\frac{\gamma}{2} = \sin\frac{\alpha}{2} \Big/ \sin\frac{\beta}{2}$$

The segment $(-\beta, \beta)$ is then deformed into the segment $(-\pi, \pi)$ and all the integrands become bounded since $d\alpha / X(\alpha) = d\gamma / 2 \cos {}^1/_2\,\alpha$. Furthermore, because of the evenness of the reaction q, and therefore, of the new desired function y also, the integrals between $-\pi$ and π can be replaced by integrals between zero and π. Only the part of the kernel R_1 and of the function Ψ_1 even in α_1 should hence be taken. We now divide the segment $0, \pi$ into N equal parts, and apply the Simpson quadrature formula. Hence in place of (3.2) we obtain a system of $N+1$ algebraic equations. The system must be computed for some one value of the parameter ω_0 defined by (1.4), since according to (3.1) - (3.4) the function y is independent of ω_0, and the reaction q and the stress resultant P are linearly independent.

4. Equation (3.2), transformed by the method mentioned above to a system of 11 algebraic equations, was solved numerically on the BESM-4 computer. The value of the function y, and the reaction q was computed for values of the coordinate $\xi_0 = 0.05, 0.1, 0.3, 0.5, 1.0, 2.0$ and 5.0. Different values of the parameter β characterizing the magnitude of the contact zone were selected for each of the ξ_0 values presented above. For $\xi_0 = 0.05$ and 0.1 we selected $\beta = 0.05, 0.1, 0.3, 0.5$ and 1.0, while for the remaining ξ_0 (except the mentioned values), we took the values $\beta = 1.5, 2.0, 2.5, 3.0$ and 3.1. Twenty terms were kept in the series for (2.7). Results for shells with the parameters $R/h = 100$ and $1 - R_0/R_1 = 0.01$ for $m = 2$ and $\xi_0 = 2$ were compared with 20 and 40 terms of the series (2.7). The first four significant figures for y hence agreed. Some results of computations for a shell with the parameters $R/h = 100$ and $1 - R_0/R_1 = 0.01$, loaded by two stamps ($m = 2$), are presented in Fig. 2.

It is shown in Fig. 2a how the solutions of (3.2) changes at the center of the arc of contact (three upper curves), and at its endpoint (three lower curves) depending on the distance ξ_0 between the shell endface and the stamp. The change in y along the arc of contact between $\alpha = 0$ and $\alpha = \beta$ is a monotonely decreasing function, convex upward. We do not present the curves $y = y(\alpha)$ for lack of space. It is seen from Fig. 2a that for small contact zones (the curve $\beta = 0.1$) the function y practically does not change up to the value $\xi_0 = 0.5$, then its value at the center of the contact zone $y(0)$ rises sharply, and the value at the endpoint $y(\beta)$ sharply decreases. As the zone of contact β increases, the character of the curves y is still retained, but the influence of the shell endface is extended farther and farther into the shell (the curves $\beta = 0.5$ and 1). For $\xi_0 = 5$ the shell behaves practically as an infinite shell, and the influence of the endface gives no effect. The picture of the change in the dimensionless reaction of the stamp $qR\beta/P$ at the center of the contact zone is shown in Fig. 2b (P is the total force applied to all the stamps from outside). The points to the right are the solution for an infinite shell. The character of the change in the dimensionless reaction along a length of contact zone of magnitude $\beta = 0.5$ is shown in Fig. 2c for different distances ξ_0 between the shell endface and the stamp. The dashes show the solution for an infinite shell obtained separately. As we see, the character of the reaction changes abruptly near the endface, but even for $\xi_0 = 0.5$ is slightly different from the reaction in an infinite shell.

In conclusion, let us mention that the unbounded growth in reaction at the endpoint of the contact zone is a corollary of hypotheses propounded in shell theory, the hypothesis of straight normals, and the hypotheses of absence of compression in the shell layers in the normal direction. Here a picture analogous to the picture in crack theory holds, when the replacement of a discrete model of a solid by a continuous model results in unbounded stresses at the crack ends. The reaction at the ends of the contact zones certainly vanishes in a real shell. However, the thicker the shell, the greater the concentration of the reaction near the ends of the contact zone. An unbounded reaction at the ends is not a disadvantage of the solution, since all the stress resultants and moments in the shell middle surface, with the exception of the transverse forces, will hence be bounded in the neighborhood of the ends of the contact zone.

5. As an application, let us present the calculation of the integral (2.11). From (1.3) we find

$$\frac{dK_1}{d\alpha} = x \operatorname{sh} 2x \frac{df_1}{d\alpha} + \frac{1}{2} f_2 \qquad (5.1)$$

$$f_1 = [\operatorname{ch} 2x - \cos(\alpha - \alpha_1)]^{-1}, \quad f_2 = \sin(\alpha - \alpha_1) / [\operatorname{ch} 2x - \cos(\alpha - \alpha_1)]$$

Let us use the notation

$$J_j = -\frac{1}{2\pi} \int_{-\beta}^{\beta} \frac{f_j(\alpha) X(\alpha)\, d\alpha}{\sin {}^1/_2 (\alpha - \alpha_0)} \qquad (j = 1, 2)$$

It can be proved that the following differentiation formula holds

$$\frac{d}{d\alpha}\left(X(\alpha_0) \int_{-\beta}^{\beta} \frac{f(\alpha)}{X(\alpha)} \frac{d\alpha}{\sin {}^1/_2 (\alpha - \alpha_0)} \right) = \frac{1}{X(\alpha_0)} \int_{-\beta}^{\beta} \frac{f'(\alpha) X(\alpha)\, d\alpha}{\sin {}^1/_2 (\alpha - \alpha_0)}$$

which is valid for an arbitrary smooth function satisfying the condition

$$X(\alpha) f(\alpha) = 0 \quad \text{при } \alpha = \pm \beta$$

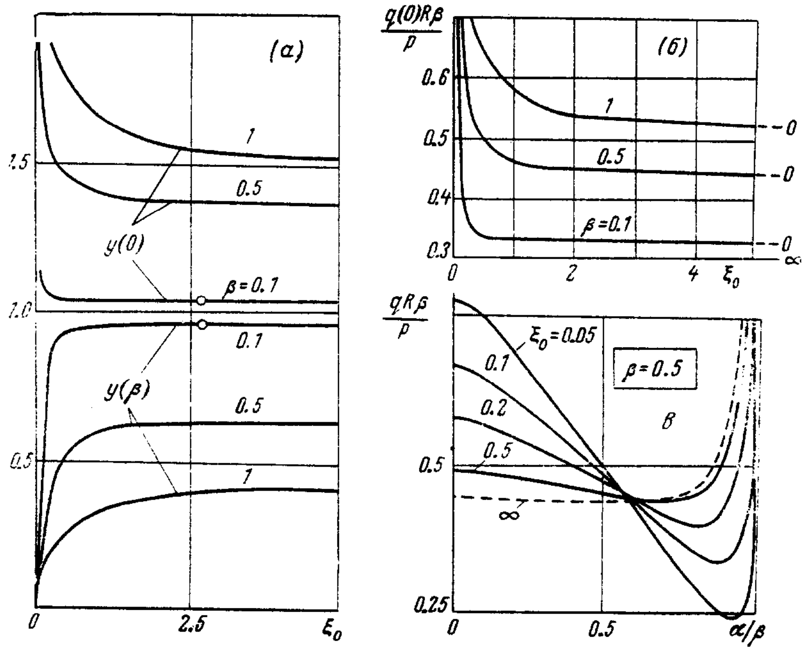

Fig. 2.

Utilizing it to evaluate the integral of the first member in (5.1), we obtain

$$R_1(\alpha_1, \alpha) = x \operatorname{sh} 2x\, X(\alpha) \frac{d}{dx} [X(\alpha) J_1(\alpha)] + \frac{1}{2} J_2(\alpha) \tag{5.2}$$

Let us evaluate $J_2(\alpha)$. Let us make the change of variable $t = \exp(i\alpha)$, which transfers the segment $(-\beta, \beta)$ into the arc of a unit circle $(-\beta, \beta)$ (Fig. 3). Let $X^+(t)$ denote the limit value of the canonical function

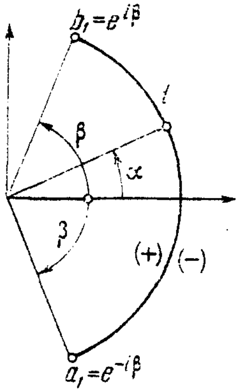

Fig. 3.

$$X(z) = \sqrt{(z - a_1)(z - b_1)}, \qquad a_1 = e^{-i\beta}, \qquad b_1 = e^{i\beta}$$

when approaching the arc t on the left if we go from a_1 to b_1. Let us understand $X(z)$ to be the branch equal to z in the neighborhood of the point $z = \infty$. Then

$$X(\alpha) = -e^{-i\frac{\alpha}{2}} X^+(t) \tag{5.3}$$

and the integral J_2 is transformed into

$$J_2 = -\frac{e^{\frac{1}{2}i\alpha_0}}{4\pi i t_1} \int_{a_1}^{b_1} \frac{(t^2 - t_1^2) X^+(t)}{t^2 [\operatorname{ch} 2x - (t^2 + t_1^2)/2tt_1]} \frac{dt}{t - t_0} \tag{5.4}$$

where and t_0 are points on an arc corresponding to α_1 and α_0. Utilizing the residue theorem and the fact that $X(z)$ changes sign upon passing from one edge of the arc (slit) to the other, we then easily obtain the following formula for integrals of the type (5.4) [5]

$$\int_{a_1}^{b_1} \frac{f(t)\, dt}{t - t_0} = -\pi i\, [\Sigma G_k(t_0) + G_\infty(t_0)] \tag{5.5}$$

where G_k and G_∞ are the principal parts of the integrand $f(z)$ at the points $z = z_k$ and $z = \infty$, respectively. We have for the integrand (4) at zero and infinity

$$G_0(t_0) = -2t_1 / t_0, \quad G_\infty(t_0) = -2t_1$$

There are still simple poles at the points

$$z_1 = t_1(\operatorname{ch}2x + \operatorname{sh}2x), \quad z_2 = t_1(\operatorname{ch}2x - \operatorname{sh}2x) \tag{5.6}$$

Substituting all four principal parts into (5.5), we obtain

$$J_2 = 2\cos\frac{\alpha_0}{2} + e^{\frac{1}{2}i\alpha_0}\left(\frac{X(z_1)}{t_0 - z_1} + \frac{X(z_2)}{t_0 - z_2}\right)$$

Now substituting z_1 and z_2 from (5.6) into the formula for $X(z)$, we find

$$X(z_1) = 2e^{\frac{1}{2}i\alpha_1}(\operatorname{ch} x + \operatorname{sh} x)\sqrt{\tfrac{1}{2}(a + ib)}, \qquad a = \operatorname{ch} 2x \cos\alpha_1 - \cos\beta$$

$$X(z_2) = 2e^{\frac{1}{2}i\alpha_1}(\operatorname{ch} x + \operatorname{sh} x)\sqrt{\tfrac{1}{2}(a - ib)}, \qquad b = \operatorname{sh} 2x \sin\alpha_1$$

Losses in sign can occur in the evaluation of $X(z_1)$ and $X(z_2)$. In order to reduce it, let us note that the point z_1 lies on a radial arc exterior to the arc of the unit radius, and z_2 on the same ray interiorly so that

$$X(z_2) \to X^+(t), \quad X(z_1) \to X^-(t) \qquad \text{for} \quad z_1 \to z_2 \to t$$

Comparing the first limit with (5.3), we see that it is necessary to change sign in the formula for $X(z_2)$. The further transformation of the integral J_2 is evident. The integral J_1 is also evaluated exactly. Performing computations, and substituting J_1, J_2 into (5.2), we arrive at (2.12). Let us evaluate the integral (2.9). Assuming $t = e^{i\alpha}$ and utilizing (5.3), we represent the integral (2.9) as

$$J_k = -\frac{t_0^{1/2}}{2\pi i}\int_{a_1}^{b_1}\frac{X^+(t)(t^k - t^{-k})}{t(t-t_0)}\,dt \tag{5.7}$$

In the neighborhood of the points $z = 0$ and $z = \infty$ the canonical function $X(z)$ can be represented, respectively, as

$$\frac{1}{X(z)} = -\frac{1}{\sqrt{1-2z\cos\beta+z^2}} = -\sum_{n=0}^{\infty} P_n z^n \tag{5.8}$$

$$\frac{1}{X(z)} = \frac{1}{z\sqrt{1-2z^{-1}\cos\beta+z^{-2}}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{P_n}{z^n}$$

where $P_n = P_n(\cos\beta)$ are Legendre polynomials. Utilizing the expansion (5.8), the principal value of the function

$$f_{(z)} = -X_{(z)}^{\cdot}(z^k - z^{-k})/z = -X^2(z)(z^k - z^{-k})/zX(z)$$

can be represented, respectively, in the neighborhood of $z = 0$ and $z = \infty$ as

$$G_0(z) = \sum_{n=0}^{k-2} P_n z^{n-k+1} - 2\cos\beta\sum_{n=0}^{k-1} P_n z^{n-k} + \sum_{n=0}^{k} P_n z^{n-k-1} \tag{5.9}$$

$$G_\infty(z) = \sum_{n=0}^{k-2} P_n z^{k-n-2} - 2\cos\beta\sum_{n=0}^{k-1} P_n z^{k-n-1} + \sum_{n=0}^{k} P_n z^{k-n}$$

Let us set $n = n_1 - 1$ in the first sum of the formula (5.9) and $n = n_1 + 1$ in the third, and again denote n_1 by n. The expressions (5.9) take the form

$$G_0(z) = \sum_{n=0}^{k-1}(P_{n+1} - 2\cos\beta\, P_n + P_{n-1})\, z^{n-k} + P_0(z^{-k-1} - z^{-k})$$

$$G_\infty(z) = \sum_{n=0}^{k-1}(P_{n+1} - 2\cos\beta\, P_n + P_{n-1})\, z^{k-n-1} + P_0(z^k - z^{k-1})$$

Let us set $z = t_0$ here, let us then substitute $G_0(t_0)$ and $G_\infty(t_0)$ into (5.5), let us then replace the subscript n by $n_1 - 1$ and again denote n_1 by n. Hence, we obtain the expression for the integral (5.7) as

$$J_k = \sum_{n=1}^{k}(P_n - 2\cos\beta P_{n-1} + P_{n-2})\cos(k-n+1/2)\,\alpha_0 + \cos(k+1/2)\,\alpha_0 - \cos(k-1/2)\,\alpha_0$$

Comparing the right side of this expression with the right side of (2.9), we obtain formula (2.10) for the coefficients a_m.

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POST-BUCKLING BEHAVIOR OF A CLOSED SPHERICAL SHELL

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The question of new equilibrium modes of a uniformly compresses closed elastic spherical shell for loading values close to the critical one is considered for which the membrane state of stress loses stability. The problem [1] reduces to constructing solutions branching off from the trivial solution in the neighborhood of the bifurcation point, for the equations in [2]. The investigation is carried out by the Liapunov-Schmidt method for a broad class of operator equations in Banach space [3].

The author of [4, 5] used the analytical Liapunov-Schmidt method earlier to construct new equilibrium modes in the case of plates and shallow shells. The problem of the bifurcation of the trivial solution of a shallow spherical segment by the Poincaré method was investigated in [6], where meridian stress resultants in equilibrium with the uniformly distributed surface pressure are given on the edge, whereupon a membrane equilibrium mode always exists. For the problem of an uniformly compressed closed sphere when the spectrum is simple, the behavior of the solutions in the neighborhood of the bifurcation point has been studied in [7] numerically on a computer by using the method of "adjustment". The survey [8] is devoted to this same problem.

1. Formulation of the problem. The Reissner equations for axisymmetric elastic deformation of a closed spherical shell subjected to uniformly distributed pressure [2] are considered in dimensionless form

$$\varepsilon^2 \Big\{(\Phi - \Phi_0)'' + \operatorname{ctg}\xi\,(\Phi - \Phi_0)' - \frac{\cos\Phi}{\sin^2\xi}(\sin\Phi - \sin\Phi_0) + \qquad (1.1)$$
$$+ \frac{\nu\Phi_0'}{\sin\xi}(\cos\Phi - \cos\Phi_0)\Big\} = \frac{1}{\sin\xi}(N\sin\Phi - T\cos\Phi)$$
$$\Big\{N'' + \operatorname{ctg}\xi\,N' - \Big(\frac{\cos^2\Phi_0}{\sin^2\xi} - \nu\Phi_0'\,\frac{\sin\Phi_0}{\sin\xi}\Big)N\Big\} = \frac{1}{\sin\xi}\{\cos\Phi - \cos\Phi_0 +$$